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THE EFFECT OF SPIN-SPIN INTERACTION ON HIGH VELOCITY SCATTERING

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ABSTRACT. The complete Mott formula of high speed electron scattering by atoms has been derived wavestatistically by assuming a spin-spin interaction potential somewhat similar to that taken by Dirac and Van Vleck in magnetism. The spin-orbit interaction is taken as in the well-known Thomas effect. The theory has been applied to electron-electron scattering. The formula thus derived agrees with Möller's formula only at the important limiting cases where the velocity in the relativistic region is (*i*) small and (*ii*) extremely high. The case of electron-positron scattering has also been considered.

In dealing with the problem of scattering of fast electrons by atoms, it was pointed out (Kar, 1946) that the complete Mott formula (Mott, 1929) of scattering, namely,

$$I=\left(\frac{Ze^2}{2m_0v^2}\right)^2\left(1-\frac{v^2}{c^2}\right)\left(\operatorname{cosec}^4\tfrac{1}{2}\theta-\frac{v^2}{c^2}\operatorname{cosec}^2\tfrac{1}{2}\theta+\pi.\frac{v}{c}.\frac{2\pi Ze^2}{hc}\frac{\cos^2\frac{1}{2}\theta}{\sin^3\frac{1}{2}\theta}\right.$$
$$\left.+\text{ higher terms}\right) \quad \ldots \quad (1)$$

could not be derived by the wavestatistical method unless some sort of spin-spin interaction was assumed between the interacting particles.

In the present paper it is proposed to derive the complete Mott formula by taking the spin-spin interaction somewhat similar to that taken by Dirac (*Principles of Quantum Mechanics*) and Van Vleck (*Electric and Magnetic Susceptibilities*) in the theory of Magnetism. The theory is also applied to electron-electron scattering and the formula thus derived is found to be in agreement with Möller's formula at very low and high velocities of incidence.

It has been shown in the previous paper that the relativistic wave equation is

$$\Delta\chi+\frac{4\pi^2}{h^2c^2}[(E-V)^2-E_0{}^2-(\tfrac{1}{2}V^2)_{s\text{-}o}]\chi=0 \qquad (2)$$

when the spin-orbit interaction potential $(\frac{1}{2}V^2)_{s\text{-}o}$ is taken into consideration. If the spin-spin interaction $(V_{s\text{-}s})$ is taken into account, the above wave equation is further modified and becomes

$$\Delta\chi+\frac{4\pi^2}{h^2c^2}[(E-V-V_{s\text{-}s})^2-E_0{}^2-(\tfrac{1}{2}V^2)_{s\text{-}o}]\chi=0 \qquad (3)$$

On proceeding in the usual manner we have, for the differential equation satisfied by the first order scattering function $(\lambda_1\chi_1)$,

$$\Delta(\lambda_1\chi_1)+k^2(\lambda_1\chi_1)=\frac{4\pi^2}{h^2c^2}\chi_0\left[2EV+2EV_{s\text{-}s}-V^2+(\tfrac{1}{2}V^2)_{s\text{-}o}\right]$$

$$k^2=\frac{4\pi^2}{h^2c^2}\left[E^2-E_0^2\right] \tag{3.1}$$

where it is assumed that the effect of the potential $V_{s\text{-}s}$ is so small that its square is negligible.

It has been shown, in the previous paper referred to above, that the first term within bracket in (3.1) gives after solution the usual cosec^4 – term in (1) whereas the third and fourth terms, namely, $-V^2$ and $(\frac{1}{2}V^2)_{s\text{-}o}$, give the third term in Mott formula (1). Let us find in the following the contribution of the second term, namely, $2EV_{s\text{-}s}$ of (3.1), in the formula for the intensity of scattering.

Now, Dirac and also Van Vleck have shown that the effective coupling between spins due to the exchange effect is equivalent to a potential energy of the form

$$V_{s\text{-}s}=-2s_1s_2J_{12} \quad \dots \quad (4)$$

where J_{12} is the exchange integral for the interacting particles 1 and 2, while s_1, s_2 are the spin angular momenta in quantum units. We shall, however, take the instantaneous value of the potential in the form

$$V_{s\text{-}s}=\pm 2\,s_1s_2\cdot\frac{Ze^2}{r} \tag{5}$$

where the upper sign denotes that the Coulomb force between the interacting particles is repulsive. In the case of electron scattering by atom nuclei as considered by Mott, the force is attractive. So the negative sign should be taken in (5).

Thus, on proceeding as before we have for the first order scattering function (Kar, 1946) due to $V_{s\text{-}s}$

$$\lambda_1\chi_1=+2s_1s_2\frac{Ze^2}{2m_0v^2}(1-\beta^2)^{\frac{1}{2}}\operatorname{cosec}^2\tfrac{1}{2}\theta\; A.\frac{e^{ikr}}{r}\cos k'r_0 \tag{6}$$

Now, the incident electron may nave the same or the opposite spin after scattering. In the former case the spin factor is evidently unity while in the latter case it is shown to be $-\cos\theta$ (Kar, *loc. cit*). Hence, the total spin factoı is

$$\delta_s=1-\cos\theta=2\sin^2\tfrac{1}{2}\theta \quad \dots \quad (7)$$

It is obvious that to include the two classes of events mentioned above, we must multiply (6) by the spin factor δ_s.

Again, we have good reasons to believe that the spin-spin interaction is effective only at high velocity of incidence. In other words, the spin-spin interaction is intimately connected with the relativity effect. Therefore the scattering function should be multiplied by yet another factor which may be called the 'relativity factor' (δ_{re}). It may be taken as

$$\delta_{re} = \frac{I_{relat} - I_{non\text{-}relat}}{I_{non\text{-}relat}} = \frac{I_{relat}}{I_{non\text{-}relat}} - 1 \qquad \ldots \quad (8)$$

It is evidently a measure of the percentage of relativistic departure of the probability of scattering. On substituting the relativistic and non-relativistic values of the intensity we find that in the case of Mott scattering

$$\delta_{re} = -\beta^2 \qquad \ldots \quad (8.1)$$

Thus from (6), (7) and (8.1) we have for the scattering function due to the spin-spin interaction

$$\lambda_1 X_1 = -4 s_1 s_2 \frac{Ze^2}{2m_0 v^2}(1-\beta^2)^{\frac{1}{2}} \beta^2 A \frac{e^{ikr}}{r} \cos k' r_0 \qquad \ldots \quad (9)$$

Hence the total scattering function from (3.1), considering the relativity effect and the spin-spin and spin-orbit interactions, would be (*vide* Kar, *l.c.*)

$$\lambda_1 X_1 = \frac{Ze^2}{2m_0 v^2}(1-\beta^2)^{\frac{1}{2}} A \frac{e^{ikr}}{r} \Big\{ \operatorname{cosec}^2 \tfrac{1}{2}\theta \cos k' r_0 - 4 s_1 s_2 \beta^2 \cos k' r_0 + (\pi - 2k' r_0)\beta \frac{\pi Z e^2}{hc} \operatorname{cosec} \tfrac{1}{2}\theta \cos^2 \tfrac{1}{2}\theta \Big\} \qquad (10)$$

Hence we have for the relative intensity of scattering

$$I = \left(\frac{Ze^2}{2m_0 v^2} \right)^2 (1-\beta^2)\Big(\operatorname{cosec}^4 \tfrac{1}{2}\theta \cos^2 k' r_0 - 8 s_1 s_2 \beta^2 \operatorname{cosec}^2 \tfrac{1}{2}\theta \cos^2 k' r_0 + (\pi - 2k' r_0)\beta \frac{2\pi Z e^2}{hc} \cos k' r_0 \frac{\cos^2 \frac{1}{2}\theta}{\sin^3 \frac{1}{2}\theta} + \text{higher terms}\Big) \qquad (11)$$

Now, for parallel spins $s_1 = +\frac{1}{2}$, $s_2 = +\frac{1}{2}$ while for antiparallel spins $s_1 = -\frac{1}{2}$, $s_2 = +\frac{1}{2}$. Also the weights for parallel and antiparallel spins are as 3:1, *i.e.*, $I = \frac{3}{4} I_1 + \frac{1}{4} I_2$.

Hence

$$I = \left(\frac{Ze^2}{2m_0 v^2} \right)^2 (1-\beta^2)\Big(\operatorname{cosec}^4 \tfrac{1}{2}\theta \cos^2 k' r_0 - \beta^2 \operatorname{cosec}^2 \tfrac{1}{2}\theta \cos^2 k' r_0 + (\pi - 2k' r_0)\beta \frac{2\pi Z e^2}{hc} \cos k' r_0 \frac{\cos^2 \frac{1}{2}\theta}{\sin^3 \frac{1}{2}\theta} + \text{higher terms}\Big) \quad \ldots \quad (12)$$

If the critical approach be neglected the above formula evidently reduces to Mott formula (1). It may be mentioned that in a recent note Kar and Sengupta (1942) have tentatively taken in Mott formula a correction factor for the critical approach, namely, $\cos^2 k' r_0$, and have found decidedly better agreement with Sengupta's (1939) experiment on electron scattering by xenon. However, the actual correction should be that given in (12) and it is evident it would give even a better agreement with the actual experiment.

ELECTRON-ELECTRON SCATTERING

In the case of electron-electron interaction the Coulomb potential is $+\frac{e^2}{r}$ and so from (5) the spin-spin interaction potential should be

$$V_{s-s} = +2s_1s_2\frac{e^2}{r} \qquad \ldots (13)$$

Now, it has been already shown by Kar and Mrs. Basu (1944) that the first order scattering function without taking into account the effect of spin-spin or spin-orbit interaction is

$$\lambda_1\chi_1' = -\frac{e^2\gamma}{2m_0c^2(\gamma-1)\gamma^*}\operatorname{cosec}^2\tfrac{1}{2}\theta^*\frac{1}{\sqrt{v}}\frac{e^{ikr}}{r}\cos k'r_0' \qquad \ldots (14.1)$$

before exchange, and after exchange

$$\lambda_1\chi_1' = -\frac{e^2\gamma}{2m_0c^2(\gamma-1)\gamma^*}\sec^2\tfrac{1}{2}\theta^*\frac{1}{\sqrt{v}}\frac{e^{ikr}}{r}\cos k''r_0'' \qquad \ldots (14.2)$$

where $\gamma = 1\Big/\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$ and $\gamma^* = \sqrt{\tfrac{1}{2}(\gamma+1)}$. Thus the relativity factor as defined in (8) should be in this case

$$\delta_r = \frac{\gamma+1}{2\gamma^2} - 1 = \frac{(1-\gamma)(1+2\gamma)}{2\gamma^2} \qquad \ldots (15)$$

As (14.1) and (14.2) give the relativistic scattering functions before and after exchange due to the Coulomb potential, the corresponding scattering functions due to the spin-spin interaction will be obviously obtained by simply multiplying (14.1) and (14.2) by the factor $2s_1s_2$. The spin factors before and after exchange are respectively

$$\left.\begin{aligned}\delta_s &= 1-\cos\theta^* = 2\sin^2\tfrac{1}{2}\theta^*\\ \delta_s &= 1+\cos\theta^* = 2\cos^2\tfrac{1}{2}\theta^*\end{aligned}\right\} \qquad (16)$$

Thus the total scattering functions before and after exchange are respectively

$$\lambda_1\chi_1' = -\frac{e^2\gamma}{2m_0c^2(\gamma-1)\gamma^*}\frac{1}{\sqrt{v}}\frac{e^{ikr}}{r}\left(\operatorname{cosec}^2\tfrac{1}{2}\theta^*\cos k'r_0' + 2s_1s_2\frac{(1-\gamma)(1+2\gamma)}{\gamma^2}\cos k'r_0'\right), \text{ and}$$

$$\lambda_1\chi_1'' = -\frac{e^2\gamma}{2m_0c^2(\gamma-1)\gamma^*}\cdot\frac{1}{\sqrt{v}}\frac{e^{ikr}}{r}\left(\sec^2\tfrac{1}{2}\theta^*\cos k''r_0'' + 2s_1s_2\frac{(1-\gamma)(1+2\gamma)}{\gamma^2}\cos k''r_0''\right) \qquad (17)$$

neglecting the spin-orbit interaction which is small because $Z=1$ in this case.

Now, for parallel spins $s_1 = +\frac{1}{2}$, $s_2 = +\frac{1}{2}$. Therefore the total scattering function, after taking into account the effect of exchange, is

$$(\lambda_1\chi_1)_1 = -\frac{e^2}{2m_0v^2}\frac{\sqrt{2(\gamma+1)}}{\gamma}\frac{1}{\sqrt{v}}\frac{e^{ikr}}{r}\Bigg[\operatorname{cosec}^2\tfrac{1}{2}\theta^*\cos k'r_0' - \sec^2\tfrac{1}{2}\theta^*\cos k''r_0''$$

$$+\frac{(1-\gamma)(1+2\gamma)}{2\gamma^2}(\cos k'r_0' - \cos k''r_0'')\Bigg] \quad \ldots \quad (18.1)$$

Again, for antiparallel spins $s_1 = +\frac{1}{2}$, $s_2 = -\frac{1}{2}$ and so the total scattering function, after taking into account the effect of exchange, is

$$(\lambda_1\chi_1)_2 = -\frac{e^2}{2m_0v^2}\frac{\sqrt{2(\gamma+1)}}{\gamma}\frac{1}{\sqrt{v}}\frac{e^{ikr}}{r}\Bigg[\operatorname{cosec}^2\tfrac{1}{2}\theta^*\cos k'r_0' + \sec^2\tfrac{1}{2}\theta^*\cos k''r_0''$$

$$-\frac{(1-\gamma)(1+2\gamma)}{2\gamma^2}(\cos k'r_0' + \cos k''r_0'')\Bigg] \quad \ldots \quad (18.2)$$

Because the ratio of intensities for parallel and antiparallel spins are as 3:1, *i.e.*, $I_1:I_2=3:1$, we have for the total relative intensity of scattering $I=\frac{3}{4}I_1+\frac{1}{4}I_2$. Hence we have from (18.1) and (18.2)

$$I=\left(\frac{e^2}{2m_0v^2}\right)^2.\frac{2(\gamma+1)}{\gamma^2}\Bigg[\operatorname{cosec}^4\tfrac{1}{2}\theta^*\cos^2k'r_0' + \sec^4\tfrac{1}{2}\theta^*\cos^2k''r_0''$$

$$-\operatorname{cosec}^2\tfrac{1}{2}\theta^*\sec^2\tfrac{1}{2}\theta^*\cos k'r_0'\cos k''r_0''$$

$$+\frac{3}{4}\frac{(1-\gamma)(1+2\gamma)}{\gamma^2}(\operatorname{cosec}^2\tfrac{1}{2}\theta^*\cos k'r_0' - \sec^2\tfrac{1}{2}\theta^*\cos k''r_0'')(\cos k'r_0' - \cos k''r_0'')$$

$$+\frac{1}{4}\frac{(\gamma-1)(2\gamma+1)}{\gamma^2}(\operatorname{cosec}^2\tfrac{1}{2}\theta^*\cos k'r_0' + \sec^2\tfrac{1}{2}\theta^*\cos k''r_0'')(\cos k'r_0' + \cos k''r_0'')$$

$$+\frac{(\gamma-1)^2(2\gamma+1)^2}{16\gamma^4}(\cos k'r_0' + \cos k''r_0'')^2 + \text{higher terms}\Bigg].$$

Because the fourth term of the above equation involves the difference of $\cos k'r_0'$ and $\cos k''r_0''$, each of which is very nearly equal to unity, the term may be neglected. Therefore

$$I=\left(\frac{e^2}{2m_0v^2}\right)^2.\frac{2(\gamma+1)}{\gamma^2}\Bigg[\operatorname{cosec}^4\tfrac{1}{2}\theta^*\cos^2k'r_0' + \sec^4\tfrac{1}{2}\theta^*\cos^2k''r_0''$$

$$-\operatorname{cosec}^2\tfrac{1}{2}\theta^*\sec^2\tfrac{1}{2}\theta^*\cos k'r_0'\cos k''r_0''$$

$$+\frac{(\gamma-1)(2\gamma+1)}{4\gamma^2}(\cos k'r_0' + \cos k''r_0'')\{\operatorname{cosec}^2\tfrac{1}{2}\theta^*\cos k'r_0'$$

$$+\sec^2\tfrac{1}{2}\theta^*\cos k''r_0'' + \frac{(\gamma-1)(2\gamma+1)}{4\gamma^2}(\cos k'r_0' + \cos k''r_0'')\} \qquad (19)$$

Special Cases.—

Case I: If the velocity is small so that $\gamma \to 1$, we have from (19)

$$I=\left(\frac{e^2}{2m_0v^2}\right)^2\cdot\frac{2(\gamma+1)}{\gamma^2}\Big[\operatorname{cosec}^4\tfrac{1}{2}\theta^*\cos^2k'r_0'+\sec^4\tfrac{1}{2}\theta^*\cos^2k''r_0''$$
$$-\operatorname{cosec}^2\tfrac{1}{2}\theta^*\sec^2\tfrac{1}{2}\theta^*\cos k'r_0'\cos k''r_0''\Big]\quad\ldots\quad(20)$$

If the critical approach be neglected, *i.e.*, $\cos k'r_0'=\cos k''r_0''=1$, eq. (20) reduces to the corresponding formula of Möller (1932).

Case II. If the velocity be great so that $\gamma >> 1$, we have from (19)

$$I=\left(\frac{e^2}{2m_0v^2}\right)^2\cdot\frac{2}{\gamma}\Big[\operatorname{cosec}^4\tfrac{1}{2}\theta^*\cos^2k'r_0'+\sec^4\tfrac{1}{2}\theta^*\cos^2k''r_0''$$
$$-\operatorname{cosec}^2\tfrac{1}{2}\theta^*\sec^2\tfrac{1}{2}\theta^*\cos k'r_0'\cos k''r_0''+\tfrac{1}{2}(\cos k'r_0'+\cos k''r_0'')\{\operatorname{cosec}^2\tfrac{1}{2}\theta^*\cos k'r_0'$$
$$+\sec^2\tfrac{1}{2}\theta^*\cos k''r_0''+\tfrac{1}{2}(\cos k'r_0'+\cos k''r_0'')\}\Big]\quad\ldots\quad(21)$$

It may be seen without difficulty that because of the factor $2/\gamma$, which is very small in the present case, the critical approach is negligible. Thus the formula is very much simplified and we have

$$I=\left(\frac{e^2}{2m_0v^2}\right)^2\cdot\frac{2}{\gamma}\Big[\operatorname{cosec}^4\tfrac{1}{2}\theta^*+\sec^4\tfrac{1}{2}\theta^*+1\Big]\quad\ldots\quad(22)$$

which is Möller's formula at high velocity

It is thus shown that the general formula (19) reduces to Möller's formulae in the two special cases considered if the critical approach is neglected. But for intermediate values of the incident velocity, lying of course in the relativistic region, (19) becomes on neglecting the critical approach,

$$I=\left(\frac{e^2}{2m_0v^2}\right)^2\cdot\frac{2(\gamma+1)}{\gamma^2}\Big[\operatorname{cosec}^4\tfrac{1}{2}\theta^*+\sec^4\tfrac{1}{2}\theta^*-\operatorname{cosec}^2\tfrac{1}{2}\theta^*\sec^2\tfrac{1}{2}\theta^*$$
$$+\frac{(\gamma-1)(2\gamma+1)}{2\gamma^2}\left\{\operatorname{cosec}^2\tfrac{1}{2}\theta^*\sec^2\tfrac{1}{2}\theta^*+\frac{(\gamma-1)(2\gamma+1)}{2\gamma^2}\right\}\Big]\quad\ldots\quad(23)$$

which is different from Möller's formula.

ELECTRON-POSITION SCATTERING

In this case the effect of exchange should not be considered. Thus we have from the first equation of (17)

$$\lambda_1\lambda_1=-\frac{e^2\gamma}{2m_0c^2(\gamma-1)\gamma^*}\frac{1}{\sqrt{v}}\frac{e^{ikr}}{r}\left(\operatorname{cosec}^2\tfrac{1}{2}\theta^*\pm\frac{(1-\gamma)(1+2\gamma)}{2\gamma^2}\right)\cos k'r_0\ldots\quad(24)$$

where the + sign is for parallel and − for antiparallel spins. Remembering that the weights for parallel and antiparallel spins are as 3:1, we have for the intensity of scattering

$$I=\left(\frac{e^2}{2m_0v^2}\right)^2\cdot\frac{2(\gamma+1)}{\gamma^2}\left[\operatorname{cosec}^4\tfrac{1}{2}\theta^*+\frac{(1-\gamma)(1+2\gamma)}{2\gamma^2}\operatorname{cosec}^2\tfrac{1}{2}\theta^*\right]\cos k'r_0\quad\ldots\quad(25)$$

neglecting higher terms. At low velocity but in the relativistic region the second term in bracket representing the spin-spin interaction is negligible and we get Bhaba's (1938) formula.

Again, if the velocity is extremely high, so that $\gamma >> 1$, we have from (25)

$$I = \left(\frac{e^2}{2m_0v^2} \right)^2 \cdot \frac{2}{\gamma} \operatorname{cosec}^2 \tfrac{1}{2}\theta^* \cot^2 \tfrac{1}{2}\theta^* \qquad (25.1)$$

neglecting the critical approach which is small.

DISCUSSION

It may be mentioned that in electron-electron and electron-positron scattering, the approximate value of the relativity factor given in (15) has been used. The rigorous value is more complicated. From the definition of δ_{re} given in (8) we should have strictly

$$\delta_{re} = \frac{\gamma + 1}{2\gamma^2} \cdot \frac{(\operatorname{cosec}^4 \frac{1}{2}\theta^*)\text{relat.}}{(\operatorname{cosec}^4 \frac{1}{2}\theta^*)\text{non-relat.}} - 1 \qquad (26)$$

In the approximate value of δ_{re} used before, the relativistic and non-relativistic values of $\operatorname{cosec}^4 \frac{1}{2}\theta^*$ have been taken approximately the same. However, on taking their exact values it may be easily shown that

$$\delta_{re} = \frac{2}{\gamma^2(\gamma+1)} \left\{ 1 + (\gamma - 1)\sin^2\theta \right\}^{-} - 1 \qquad (27)$$

If the velocity is too small so that $\gamma \rightarrow 1$, then $\delta_{re} \rightarrow 0$ as in the approximation. Again, if the velocity is very great so that $\gamma >> 1$, $\delta_{re} \rightarrow -1$ as before. Thus the limiting values of δ_{re} rigorously evaluated are exactly the same as in the approximations used before.

Lastly, the question of correction for the critical approach has not been discussed in the present paper as it is too small. The uncorrected value may be used.

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